1. (a) The kinetic energy of a particle is $E = \frac{1}{2}mv^2$. Evaluate $\frac{dE}{dv}$.

$$\frac{dE}{dv} = mv$$

(b) A particle has velocity $v(t) = Ae^{-bt}$ and kinetic energy $E(t) = \frac{1}{2}mv(t)^2$. Evaluate $\frac{dE}{dt}$.

$$\frac{dE}{dt} = mv\frac{dv}{dt} = m(Ae^{-bt})(-bAe^{-bt}) = -mbA^2e^{-2bt}$$

(c) The partition function for a system is $q(T) = T^2e^{-\epsilon/k_BT}$. Evaluate $\frac{dq}{dT}$.

$$\frac{dq}{dT} = 2Te^{-\epsilon/k_BT} + T^2e^{-\epsilon/k_BT}\left(\frac{\epsilon}{k_BT^2}\right) = e^{-\epsilon/k_BT}\left(2T + \frac{\epsilon}{k_B}\right)$$

(d) Evaluate $\int_0^\infty ve^{-mv^2/2k_BT}dv$. You may use the following integral: $\int_0^\infty e^{-x}dx = 1$

$$u = \frac{mv^2}{2k_BT} \implies du = \frac{mv}{k_BT}dv \implies dv = \frac{k_BT}{mv}du$$

$$\int_0^\infty ve^{-mv^2/2k_BT}dv = \frac{k_BT}{m}\int_0^\infty e^{-u}du = \frac{k_BT}{m}$$

(e) The ideal gas law is $PV = nRT$. Solve for P in terms of V and T, then evaluate $\left(\frac{\partial P}{\partial V}\right)_{T,n}$.

$$P = \frac{nRT}{V} \implies \left(\frac{\partial P}{\partial V}\right)_{T,n} = -\frac{nRT}{V^2}$$

(f) The energy of ideal gas is $U = \frac{3}{2}nRT$. Without substituting, calculate $\left(\frac{\partial U}{\partial T}\right)_V$ and $\left(\frac{\partial U}{\partial V}\right)_T$.

$$\left(\frac{\partial U}{\partial T}\right)_V = \frac{3}{2}nR \qquad \left(\frac{\partial U}{\partial V}\right)_T = 0$$

(g) For the ideal gas $P = nRT/V$, compute $\left(\frac{\partial P}{\partial V}\right)_{n,T}$ and $\left(\frac{\partial P}{\partial T}\right)_{n,V}$.

$$\left(\frac{\partial P}{\partial V}\right)_T = -\frac{nRT}{V^2} \qquad \left(\frac{\partial P}{\partial T}\right)_V = \frac{nR}{V}$$

2. The internal energy U is defined as $U = k_B T^2 \left(\frac{\partial \ln Q}{\partial T}\right)_{N,V}$ where k_B is the Boltzmann constant, and the partition function Q is defined as $Q(N, V, T) = \frac{1}{N!}\left(\frac{2\pi m k_B T}{h^2}\right)^{3N/2} V^N$, where m is the mass of the particle. Determine U as a function of T.

$$Q = \frac{1}{N!}\left(\frac{2\pi m k_B T}{h^2}\right)^{3N/2} V^N$$

$$\ln Q = -\ln N! + \frac{3N}{2}\ln(2\pi m k_B T/h^2) + N \ln V$$

$$\left(\frac{\partial \ln Q}{\partial T}\right)_{N,V} = \frac{3N}{2}\frac{1}{T}$$

$$U = k_B T^2 \left(\frac{3N}{2}\frac{1}{T}\right) = \boxed{\frac{3}{2}N k_B T}$$

3. Starting from the ideal gas law, prove that $\left(\frac{\partial P}{\partial V}\right)_{n,T} = \frac{1}{\left(\frac{\partial V}{\partial P}\right)_{n,T}}$ by evaluating both sides.

The ideal gas law is $PV = nRT$, so we isolate each variable and take the partial.

$$P = \frac{nRT}{V} \implies \left(\frac{\partial P}{\partial V}\right)_{n,T} = -\frac{nRT}{V^2} = LHS$$

$$V = \frac{nRT}{P} \implies \left(\frac{\partial V}{\partial P}\right)_{n,T} = -\frac{nRT}{P^2}$$

$$RHS = \frac{1}{\left(\frac{\partial V}{\partial P}\right)_{n,T}} = -\frac{P^2}{nRT} = -\frac{(nRT)^2/V^2}{nRT} = -\frac{nRT}{V^2}$$

$RHS = LHS$, thus proving $\left(\frac{\partial P}{\partial V}\right)_{n,T} = \frac{1}{\left(\frac{\partial V}{\partial P}\right)_{n,T}}$

Homework Problem 1

1. Consider an ideal gas ($PV = nRT$), where the internal energy is given by $U = \frac{3}{2}nRT$.
   (a) Express T in terms of U and n.

   (b) Write P as a function of U and V.

   (c) Compute the following partial derivatives:

$$\left(\frac{\partial P}{\partial U}\right)_V \qquad \left(\frac{\partial U}{\partial V}\right)_P \qquad \left(\frac{\partial U}{\partial P}\right)_V$$

   (d) What do you notice about $\left(\frac{\partial P}{\partial U}\right)_V$ and $\left(\frac{\partial U}{\partial P}\right)_V$?