1. Consider a particle that is constrained to lie along a 1D segment 0 to a. The probability of finding this particle is given by $p(x)\,dx = \frac{2}{a}\sin^2\left(\frac{n\pi x}{a}\right)dx$. Find the probability that the particle will be found between 0 and $a/2$. You will need the following integral: $\int \sin^2(kx)\,dx = \frac{x}{2} - \frac{\sin(2kx)}{4k}$

$$
\begin{aligned}
P\left(0 \le x \le \frac{a}{2}\right) &= \int_0^{a/2} p(x)\,dx = \int_0^{a/2} \frac{2}{a}\sin^2\left(\frac{n\pi x}{a}\right)dx \\
&= \frac{2}{a}\left[\frac{x}{2} - \frac{\sin\left(2\frac{n\pi}{a}x\right)}{4(n\pi/a)}\right]_0^{a/2} \\
&= \frac{2}{a}\left(\left[\frac{a/2}{2} - \frac{\sin\left(2\frac{n\pi}{a}\cdot a/2\right)}{4(n\pi/a)}\right] - \left[0 - \frac{\sin(0)}{4(n\pi/a)}\right]\right) \\
&= \frac{2}{a}\left(\frac{a}{4} - \cancel{\frac{\sin(n\pi)}{4(n\pi/a)}} - 0 + 0\right) \\
&= \frac{2}{a}\cdot\frac{a}{4} = \boxed{\frac{1}{2}}
\end{aligned}
$$

2. Consider a system with 5 energy levels (E_1, E_2, E_3, E_4, E_5) and 3 particles distributed among these levels. Each energy level can hold any number of particles.

   (a) How many microstates (arrangements of particles) are possible if the particles are distinguishable?

$$5 \times 5 \times 5 = 125$$

   (b) How many microstates are possible if the particles are indistinguishable?

   Using stars and bars method:

$$\binom{7}{3} = \frac{7!}{3!\cdot 4!} = 35$$

   (c) How many microstates are possible if the particles obey the Pauli Exclusion Principle, meaning each level can only hold one particle?

$$\binom{5}{3} = \frac{5!}{3!\cdot 2!} = 10$$

3. Three atoms in a crystal lattice oscillate with quantized energies given by: $E_n = h\nu\left(n + \frac{1}{2}\right)$, where $n \geq 0$ is the quantum number. The total energy of the system is: $E_{total} = h\nu\left(n_1 + n_2 + n_3 + \frac{3}{2}\right)$ where n_1, n_2, and n_3 are the quantum numbers of the three atoms.

   It is observed that the total energy of the system is $\frac{9}{2}h\nu$. Determine the most likely configuration of the quantum numbers n_1, n_2, and n_3 and the probability of obtaining this configuration.

   The total energy is:

$$E_{\text{total}} = h\nu\left(n_1 + n_2 + n_3 + \frac{3}{2}\right) = \frac{9}{2}h\nu.$$

   Therefore:

$$n_1 + n_2 + n_3 + \frac{3}{2} = \frac{9}{2} \quad \Rightarrow \quad n_1 + n_2 + n_3 = 3.$$

   There are three combinations of positive integers that sum to 3, with the following number of permutations:

$$(3,0,0):3 \qquad (2,1,0):6 \qquad (1,1,1):1$$

   There are 10 total permutations, each with equal probability, so the most likely configuration is $\boxed{(2,1,0)}$ with probability $\boxed{\frac{3}{5}}$.

4. The probability of finding a particle at position x is given by $f(x) = \frac{2x}{(1+x^2)^{3/2}}, \quad x \geq 0$. Find the probability that x lies between 0 and 1. $P(0 \leq x \leq 1)$.

$$\begin{aligned}
P\left(0 \leq x \leq 1\right) = \int_0^1 f(x)\,dx &= \int_0^1 \frac{2x}{(1+x^2)^{3/2}}\,dx \\
\text{Let } u = 1 + x^2 \quad &\Longrightarrow \quad du = 2x\,dx \\
&= \int_{u=1}^{u=2} u^{-3/2}\,du \\
&= \left[-2\,u^{-1/2}\right]_1^2 \\
&= -2\left(\frac{1}{\sqrt{2}} - 1\right) \\
&= 2\left(1 - \frac{1}{\sqrt{2}}\right) \\
&= \boxed{2 - \sqrt{2}}
\end{aligned}$$

Homework Problem 2

1. In the kinetic theory of gases, gas molecules travel at different speeds, and the probability of finding a molecule with speed v is given as

$$p(v)\,dv = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-\frac{mv^2}{2k_B T}}\,dv$$

Show that this probability distribution is normalized. You may use the following integral:

$$\int_0^\infty x^{2n} e^{-\alpha x^2} dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}\alpha^n}\sqrt{\frac{\pi}{\alpha}}$$