1. For the equation:

$$f(v_x) = Ae^{-\gamma v_j^2}$$

determine whether γ must be strictly positive, strictly negative, or either by considering limits.

As speed $v_x \to \pm\infty$ $f(v_x) \to 0$ since the probability must be finite. If $\gamma < 0$, then exponent is positive, and the limit diverges. Therefore $\gamma > 0$

2. Without looking at the slides for the result, evaluate $\langle v_x + \rangle$ and $\langle v_x \rangle$. $\langle v_x + \rangle$ denotes the average of positive values of v_x. Potentially useful integrals:

$$f(v_x) = \left(\frac{M}{2\pi RT}\right)^{1/2} e^{-Mv_x^2/2RT}$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}} \qquad \int_{-\infty}^{\infty} x^2 e^{-ax^2}\,dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}} \qquad \int_0^{\infty} xe^{-ax^2}\,dx = \frac{1}{2a} \quad \text{for } a > 0$$

$$\begin{aligned}
\langle v_x^+ \rangle &= \int_0^{\infty} v_x\, f(v_x)\, dv_x = \left(\frac{M}{2\pi RT}\right)^{1/2} \int_0^{\infty} v_x\, e^{-Mv_x^2/2RT}\, dv_x \qquad \left(a = \frac{M}{2RT}\right) \\
&= \left(\frac{M}{2\pi RT}\right)^{1/2} \cdot \frac{1}{2\left(\frac{M}{2RT}\right)} = \left(\frac{M}{2\pi RT}\right)^{1/2} \cdot \frac{RT}{M} \\
&= \left(\frac{RT}{2\pi M}\right)^{1/2} \\
\langle v_x \rangle &= \int_{-\infty}^{\infty} v_x\, f(v_x)\, dv_x = \left(\frac{M}{2\pi RT}\right)^{1/2} \int_{-\infty}^{\infty} v_x\, e^{-Mv_x^2/2RT}\, dv_x = 0
\end{aligned}$$

v_x is odd function, e^{-x^2} is an even function. Odd × even is odd, and integral of odd is 0.

3. For the speed distribution, show that $\langle u \rangle = \sqrt{\frac{8RT}{\pi M}}$. Potentially useful integral:

$$\int_0^{\infty} x^3 e^{-ax^2}\,dx = \frac{1}{2a^2}$$

$$\langle u \rangle = \int_0^{\infty} u\, F(u)\, du = 4\pi \left(\frac{M}{2\pi RT}\right)^{3/2} \int_0^{\infty} u^3\, e^{-Mu^2/2RT}\, du \qquad \left(a = \frac{M}{2RT}\right)$$

$$\int_0^{\infty} u^3 e^{-au^2}\, du = \frac{1}{2a^2} = \frac{1}{2}\left(\frac{2RT}{M}\right)^2$$

$$\begin{aligned}
\langle u \rangle &= 4\pi \left(\frac{M}{2\pi RT}\right)^{3/2} \cdot \frac{1}{2}\left(\frac{2RT}{M}\right)^2 \\
&= 2\pi \cdot M^{-1/2} \cdot (2RT)^{1/2} \cdot (2\pi)^{-3/2} \\
&= \sqrt{\frac{8RT}{\pi M}}
\end{aligned}$$

4. (a) Derive an expression for u_{probable}.

$$\frac{dF}{du} = 0$$

The constant terms disappear when we take the derivative and set to 0

$$\begin{aligned}
F(u) &= \cancel{4\pi} u^2 \cancel{\left(\frac{M}{2\pi RT}\right)^{3/2}} e^{-Mu^2/2RT} \\
&= u^2 e^{-Mu^2/2RT} \\
\frac{dF}{du} &= 2u\, e^{-Mu^2/2RT} + u^2\left(-\frac{M}{RT}\right) u\, e^{-Mu^2/2RT} \\
\frac{dF}{du} &= u\, e^{-Mu^2/2RT}\left(2 - \frac{Mu^2}{RT}\right) \\
0 &= 2 - \frac{Mu_p^2}{RT} \implies \boxed{u_{\text{probable}} = \sqrt{\frac{2RT}{M}}}
\end{aligned}$$

(b) Rank the characteristic speeds u_{rms}, u_{probable} and $\langle u \rangle$ in ascending order.

$$u_{\text{probable}} = \sqrt{\frac{2RT}{M}} < \langle u \rangle = \sqrt{\frac{8RT}{\pi M}} < u_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

(c) Compute u_{rms}, u_{probable} and $\langle u \rangle$ numerically for N_2 at 298 K and rank them in ascending order. Does this match our expectations?

Watch for units on molar mass!

$$M_{N_2} = 28.02 \times 10^{-3}\,\frac{\text{kg}}{\text{mol}} \quad R = 8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}} \quad T = 298\,\text{K}$$

$$\begin{aligned}
u_{\text{probable}} &= \sqrt{\frac{2(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}})(298\,\text{K})}{0.02802\,\frac{\text{kg}}{\text{mol}}}} = 421\,\frac{\text{m}}{\text{s}} \\
\langle u \rangle &= \sqrt{\frac{8(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}})(298\,\text{K})}{\pi(0.02802\,\frac{\text{kg}}{\text{mol}})}} = 475\,\frac{\text{m}}{\text{s}} \\
u_{\text{rms}} &= \sqrt{\frac{3(8.314\,\frac{\text{J}}{\text{mol}\cdot\text{K}})(298\,\text{K})}{0.02802\,\frac{\text{kg}}{\text{mol}}}} = 515\,\frac{\text{m}}{\text{s}}
\end{aligned}$$

$$421\,\frac{\text{m}}{\text{s}} < 475\,\frac{\text{m}}{\text{s}} < 515\,\frac{\text{m}}{\text{s}} \implies u_p < \langle u \rangle < u_{\text{rms}} \checkmark$$

Homework Problem 31

1. Compute $\langle v_x^2 \rangle$ to verify the result from lecture, $\langle v_x^2 \rangle = \frac{RT}{M}$.

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}\, dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$$